Vector Spaces

1. Definition:

A vector space is a set of vectors together with a rule for adding any 2 vectors, v and w, to produce a vector $v+w$ in V and a rule for multiplying any vector v in V by any scalar r in $\mathbb{R}$ to produce a vector rv in V.

Furthermore, a vector space must satisfy these conditions:

Let $u, v, w \in V$ and let $r, s \in \mathbb{R}$

Properties of Vector Addition

A1. $(u+v)+w = u+(v+w)$ (Associative)
A2. $u+v = v+u$ (Commutative)
A3. $0+v = v$ (0 as an additive identity)
A4. $v+(-v) = 0$ ($-v$ as an additive inverse)

Properties of Scalar Multiplication

S1. $r(v+w) = rv + rw$ (Distributive)
S2. $(r+s)v = rv + rs$ (Distributive)
S3. $r(sv) = (rs)v$ (Associative)
S4. $1v = v$ (Preservation of Scale)

To prove something is a vector space, you have to prove that it satisfies all 8 conditions.

2. Properties:

Every vector space, V, has the following properties:-

1. The vector 0 is the unique vector x satisfying the eqn $x+v=v$, $\forall v \in V$.

2. For each vector v in V, the vector $-v$ is the unique vector satisfying the eqn $v+y=0$.

3. If $u+v=u+w$, for vectors $u, v, w \in V$, then $v=w$.

4. $0v = 0, \forall v \in V$

5. $r0 = 0, \forall r \in R$

6. $(-r)v = -(rv) = r(-v), \forall r \in R, v \in V$

3. Examples:

1. Let F be the set of all real-valued functions with domain R. Let addition be defined as $(f+g)(x) = f(x) + g(x)$. Let multiplication be defined as $(rf)(x) = rf(x)$. Show that F, with its operations.

Proof:

A1. $(f+g)(x) + h(x) = f(x) + g(x) + h(x)$
$= f(x) + (g+h)(x)$

A2. $f(x) + g(x) = g(x) + f(x)$

A3. $0 + f(x) = f(x)$

A4. $f(x) + (-f)(x) = f(x) + (-1)f(x)$
$= f(x) - f(x)$
$= 0$

S1. $(r(f+g)(x)) = (rf)(x) + (rg)(x)$
$= rf(x) + rg(x)$

S2. $(r+s)f(x) = rf(x) + sf(x)$

S3. $r(sf(x)) = rsf(x)$
$= (rs)f(x)$

S4. $(1f(x)) = 1f(x)$
$= f(x)$